BASIC EQUATIONS IN DIFFERENTIAL FORMS

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1. Reynolds Transport Theorem

Consider the flow through a pipe whose cross-sectional area changes in the flow direction. Assume that the CV is stationary.

the outflow from the CV from t to $t + \delta t$ = II

the inflow from t to t + δt = I

the system at t = CV

the system at $t+\delta t$ = CV – I + II

For the total of any extensive property B such as mass, momentum, or energy, the Reynolds transport theorem is given by

$$\left.\frac{DB}{Dt}\right|_{SYS} = \left.\frac{\partial B}{\partial t}\right|_{CV} + \sum \rho_0 A_0 V_0 b_0 - \sum \rho_i A_i V_i b_i \qquad (1)$$

where

$$B = mb$$

in which b represents the amount of B per unit mass. The Reynolds transport theorem states that the instantaneous rate of change of B in the system is the rate of accumulation of B in the CV plus the net rate out of B from the CV. Thanks to the Reynolds transport theorem, we can compute $DB/Dt|_{SYS}$ (the time rate of change of B in the system) in terms of $\partial B/\partial t|_{CV}$ (the time rate of change of B in the control volume) and fluxes in and out.

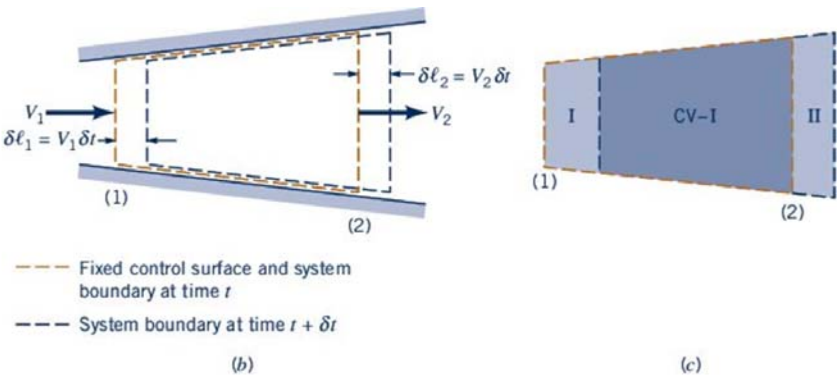

Figure 1. Control volume and system for flow through a pipe changing cross sectional area

2. Continuity Equation: conservation of mass

The continuity equation can be derived using the Reynolds transport theorem by setting $B = M$. Since the mass within the system does not change with time, we have

$$\left.\frac{DM}{Dt}\right|_{SYS} = 0 \tag{2}$$

or

$$\left.\frac{\partial M}{\partial t}\right|_{CV} + \sum \rho_o A_o V_o - \sum \rho_i A_i V_i = 0 \tag{3}$$

which can be rewritten as

$$\left.\frac{\partial M}{\partial t}\right|_{CV} = \sum \rho_i A_i V_i - \sum \rho_o A_o V_o \tag{4}$$

which indicates that the time rate of change of the mass within the CV is the net mass flux in. The term on the LHS of Eq.(4) is the time rate of change of the mass within the CV, which is given by

$$\left.\frac{\partial M}{\partial t}\right|_{CV} = \frac{\partial \rho}{\partial t}\delta x \delta y \delta z \tag{5}$$

The first term on the RHS of Eq.(4) is the mass flux going into the CV during δt. In the three-dimensional space, this is given by

$$\rho u \delta y \delta z + \rho v \delta x \delta z + \rho w \delta x \delta y \tag{6}$$

The second term on the RHS of Eq.(4) is the mass flux coming out of the CV during δt, which is

$$\left(\rho u + \frac{\partial \rho u}{\partial x}\delta x\right)\delta y \delta z + \left(\rho v + \frac{\partial \rho v}{\partial y}\delta y\right)\delta x \delta z + \left(\rho w + \frac{\partial \rho w}{\partial z}\delta z\right)\delta x \delta y \tag{7}$$

Thus, the net mass flux in is given by

$$-\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right)\delta x \delta y \delta z \tag{8}$$

Therefore, we can obtain the equation for the mass conservation or continuity equation by equating Eq.(5) and Eq.(8). That is,

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right) \tag{9}$$

If the density is constant (the fluid is incompressible), then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10}$$

Note that this equation is valid whether the velocity is time dependent or not.

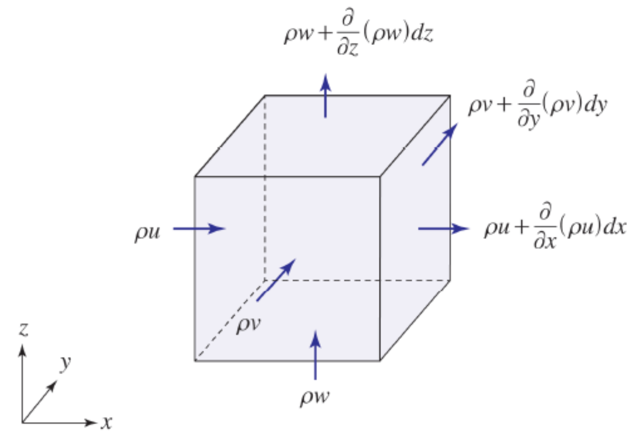

Figure 2. A differential element for mass conservation

(Q) Eq.(9) or Eq.(10) is called as the differential form of continuity equation. What is the integral form of the continuity equation?

3. Streamlines and Stream Function

The streamline is a curve everywhere parallel to the direction of the flow, whereas the pathline is a trajectory of a single particle of fluid. The streakline is a line which is traced out by a neutrally buoyant fluid that is continuously injected into a flowfield at a fixed point.

For steady or unsteady flow, mathematical definition of streamline may be given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{11}$$

where *dx*, *dy*, and *dz* are the elements of streamline segment and *u*, *v*, and *w* are corresponding velocity components. (Can you derive eq?)

Tokaty (1971)

Euler got the most beautiful answer directly or by implication. Let's imagine a continuous curved line *l*, within a fluid flow, at any given instance of time tangential to velocity vectors of all fluid particles through which it passes, the so-called streamline. The word "tangential" implies that anywhere along the streamline the velocity vector is parallel to the portion of *l* where it acts. Euler exploited this fact in a somewhat complicated way; but if we apply to it the theorem that the vector product of two parallel vectors is zero, we have

$$\vec{v} \times dl = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0$$

Since the unit vectors are $\vec{i} \neq 0$, $\vec{j} \neq 0$, and $\vec{k} \neq 0$, Eq.(11) can be directly obtained.

The concept of streamlines can be related to the continuity equation by using the stream function. For two-dimensional incompressible fluid, the continuity equation can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{12}$$

If one introduces a new function defined by the following equations:

$$u=\frac{\partial \psi}{\partial y} \tag{13}$$

$$v=-\frac{\partial \psi}{\partial x} \tag{14}$$

where *y* is a stream function which is a function of space and time. It is apparent that the stream function satisfies the continuity equation automatically. Using the total derivative of the stream function,

$$d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy$$

$$= -vdx+udy \tag{15}$$

If $d\psi$ is set to zero, the equation of streamline is obtained again. That is, lines of constant ψ represent streamlines.

(Q) Explain the relationship between equation of streamline, Eq.(15), and the stream function.

4. Momentum Equation for Inviscid Fluids

Let *B* be the linear momentum ***P***. That is,

$$\mathbf{P} = \int_{SYS} \mathbf{V} dm \tag{16}$$

Then, for the linear momentum ***P***, the Reynolds transport theorem becomes

$$\left.\frac{D\mathbf{P}}{Dt}\right|_{SYS} = \frac{\partial}{\partial t}\int_{CV} \mathbf{V}\rho d\forall + \sum \mathbf{V}_o \rho_o A_o V_o - \sum \mathbf{V}_i \rho_i A_i V_i \tag{17}$$

where the LHS term is given by

$$\left.\frac{D\mathbf{P}}{Dt}\right|_{SYS} = \mathbf{F} \tag{18}$$

which is the result from the Newton's second law.

Here, we have two options. One is that we apply Eq.(18) to a small fluid element δm, and the other is that we apply Eq.(17) to an infinitesimal CV, which initially bounds the mass, δm. It is interesting to note that we have the same result if we apply Eq.(17) to δm (White, 2003). In the present chapter, we adopt the former approach. If we apply Eq.(18) to a small fluid element δm, we have

$$\delta \mathbf{F} = \delta m \frac{D\mathbf{V}}{Dt} = \delta m \mathbf{a} \tag{19}$$

which is simply the Newton's second law.

4.1 1D Steady-State Euler Equation

Consider the 1D flow along the streamtube. Assume that the length of the streamtube (*ds*) is enough short for the cross sectional area (dA) not to change. If the fluid viscosity is ignored, then the pressure force and the body force constitute the external force. Then, in the direction of the flow, the external force $\delta \mathbf{F}$ is

$$\delta \mathbf{F} = pdA - \left(p + dp\right) dA - g\delta m \sin\theta = -dpdA - g\delta m \sin\theta \quad (20)$$

where $\sin\theta = dz / ds$ and the fluid mass within the streamtube is given by

$$\delta m = \rho dA\, ds \quad (21)$$

Thus, Eq.(20) becomes

$$\delta \mathbf{F} = -dpdA - \rho g dA dz \quad (22)$$

The acceleration a_s in Eq.(19) is given by

$$a_s = \frac{dV}{dt} + V\frac{dV}{ds} \quad (23)$$

For steady flows, $a_s = VdV / ds$. By equating the LHS and the RHS of Eq.(19), we have

$$(\rho ds dA)V\frac{dV}{ds} = -dp\; dA - \rho g dA\; dz \quad (24)$$

Divide the above equation by $\rho g dA$ results

$$d\left(\frac{V^2}{2g}\right) + \frac{dp}{\gamma} + dz = 0 \quad (25)$$

If the density of fluid constant, then

$$d\left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right) = 0 \quad (26)$$

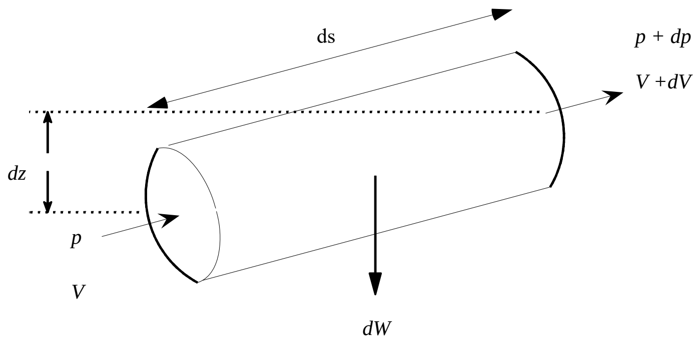

Figure 3. 1D flow along a streamtube

4.2 3D Euler Equations

Euler equations can directly be derived from Navier-Stokes equations by ignoring the viscous terms. (If inertial terms are ignored, Stokes equations are obtained for flows at very small Reynolds number). In a view point, the Euler equations are more useful for turbulent flows where the turbulent viscosities are dominant over fluid viscosity. The viscous terms for the eddy viscosity are encountered by averaging the Euler equations over turbulence.

(Q) Assuming that a fluid is incompressible, demonstrate the Euler equation for an inviscid fluid such as

$$\rho \frac{DV}{Dt} = -\nabla(p + \rho g z)$$

is also applicable to a viscous fluid if the flow is irrotational. Why is the acceleration term conservative?

Consider the cubic fluid element below. We will use Newton's 2nd law to derive the 3D Euler equations. The shear stresses are ignored because the fluid is assumed to be inviscid. Similarly, the external force includes the body force and pressure force. In the *x*-direction, if the *i*-component body force per unit mass is denoted by f_i, then the left hand side of Eq.(19) is given by

$$\delta \mathrm{F}_x = pdydz - \left(p + \frac{\partial p}{\partial x} dx \right) dydz + \rho f_x dxdydz = -\frac{\partial p}{\partial x} dxdydz + \rho f_x dxdydz \qquad (27)$$

and the RHS of Eq.(19) is

$$\delta m a_x = (\rho dxdydz) \frac{du}{dt} \qquad (28)$$

where *u* depends upon space and time. The total change in *u* between two locations shown below can be written as

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \qquad (29)$$

where $dx = udt$, $dy = vdt$, and $dz = wdt$. Then, with the help of Eq.(29), the acceleration in the *x*-direction is

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \qquad (30)$$

The derivative given by Eq.(30) is the change in *u* as the particle moves with the fluid. This *derivative following the fluid is called as total derivative or material derivative*, sometimes expressed by using such an operator as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{31}$$

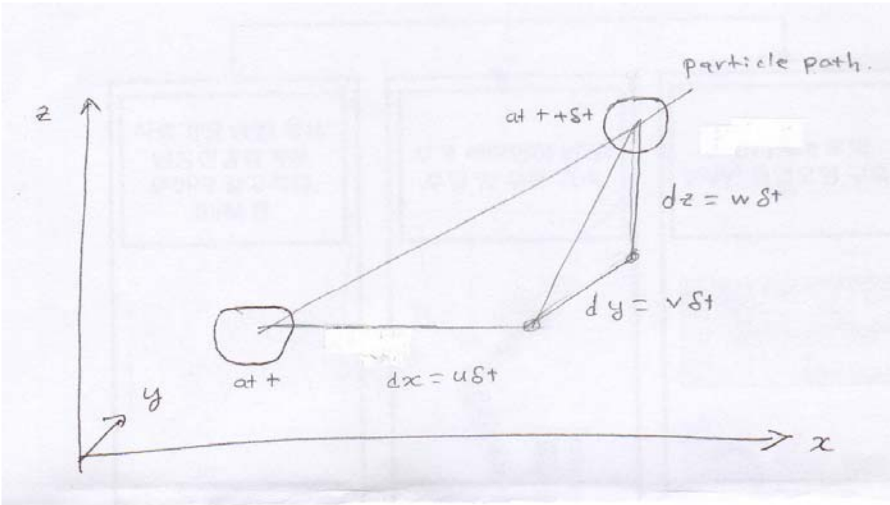

Figure 3. 3D trajectory of a fluid particle

Therefore, in the *x*-direction, the momentum equation is

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_x \tag{32a}$$

Similarly, the momentum equations in the *y*- and *z*-directions are given, respectively, by

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + f_y \tag{32b}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + f_z \tag{32c}$$

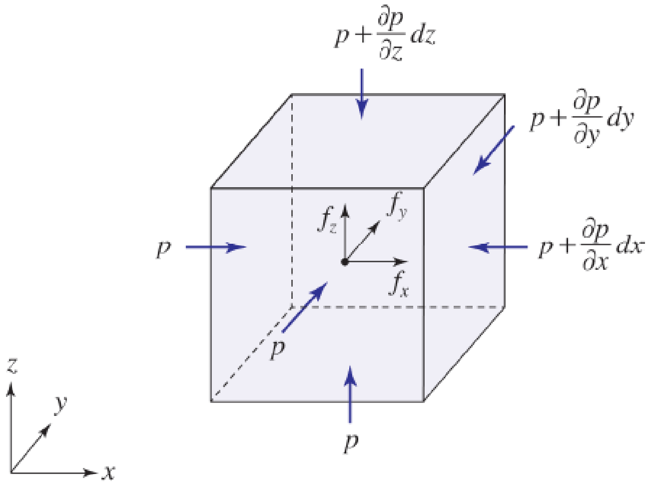

Figure 5. External forces on cubic element

Eqs.(32) were derived by Euler in 1755, so they are called the Euler equations. Since no assumptions were made about density, the equations are valid for both compressible and incompressible fluid.

As was done herein, the Euler equations can be derived using Newton's 2nd law, which was proposed around 1687. They look the same as Newton's 2nd law, if you look them closely. However, the Euler obtained the equations independently with the findings of Isaac Newton. This is why we call Euler equation to commemorate his achievement.

The Euler equations constitute a hyperbolic system of partial differential equations (without any dissipation or viscous terms). Since the number of unknowns (u, v, w, and p) is the same as the number of equations (Euler equations + continuity equation), the equations can be solved mathematically without making any further approximations.

Lamb (1879)

To calculate the rate at which any function F(x, y, z, t) varies for a moving particle, we may remark that at the time $t+\delta t$ the particle which was originally in the position (x, y, z) is in the position ($x+u\delta t, y+v\delta t, z+w\delta t$), so the corresponding value of F is

$$F\left(x+u\delta t, y+v\delta t, z+w\delta t, t+\delta t\right)=F+u\delta t\frac{\partial F}{\partial x}+v\delta t\frac{\partial F}{\partial y}+w\delta t\frac{\partial F}{\partial z}+\delta t\frac{\partial F}{\partial t}$$

Let D/Dt be a differentiation following the motion of the fluid, the new value of F is expressed by $F+DF/Dt\cdot\delta t$, whence

$$\frac{DF}{Dt}=\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}$$

5. Hydrostatics

When the fluid velocity is zero, the vector form of Eqs.(32) becomes

$$\frac{1}{\rho}\nabla p=f \tag{33}$$

Consider the gravity force is the only body force, that is,

$$f_x=f_y=0 \qquad \text{and} \qquad f_z=-g \tag{34}$$

where z is taken positively upward from the surface of the ground. Then the integration of Eq.(33) in the z-direction yields

$$p=-\rho gz+\text{constant} \tag{35}$$

Eq.(35) describes the pressure decreases linearly with increasing height, i.e., hydrostatic pressure distribution.

6. Bernoulli Equation

Consider the steady two-dimensional (x-z) flow of an incompressible, inviscid fluid,

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+f_x \tag{36}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+f_z \tag{37}$$

Using the definition of the streamline, the elements dx and dz are related to the velocity components by

$$\frac{dz}{dx}=\frac{w}{u} \tag{38}$$

Then, with the help of Eq.(38), Eqs.(36) and (37) can be combined into one equation such as

$$d\left(\frac{p}{\gamma}+\frac{V^2}{2g}+z\right)=0 \tag{39}$$

where

$$V^2=u^2+w^2 \tag{40}$$

$$dF=f_x dx+f_z dz \tag{41}$$

Eq.(41) denotes the total differential of a force potential, and the force having a potential is said to be a conservative force. Integrating Eq.(39) yields

$$\frac{V^2}{2}+\frac{p}{\rho}-F=\text{constant} \tag{42}$$

For unsteady flows, we have to add $\left(\partial u/\partial t\right)dx$ for the *x*-direction and $\left(\partial w/\partial t\right)dz$ for the *z*-direction, respectively. But these are components of $\left(\partial V/\partial t\right)ds$, where ds is an element of the streamline. Thus

$$d\left[\int_0 \frac{\partial V}{\partial t}ds+\frac{V^2}{2}+\frac{p}{\rho}-F\right]=0 \tag{43}$$

Notice that the integration of the unsteady term is carried out along a given streamline at a given instant of time from an arbitrary reference point. The integration of Eq.(43) yields a non-zero constant in the right-hand side of the equation, which is called the Bernoulli constant. The constant, which is a function of time, has a certain value for a fixed streamline. Integrating the above equation between two points of streamline leads to

$$\int_0^1 \frac{\partial V}{\partial t}ds+\frac{V_1^{\,2}}{2}+\frac{p_1}{\rho}-F_1=\int_0^2 \frac{\partial V}{\partial t}ds+\frac{V_2^{\,2}}{2}+\frac{p_2}{\rho}-F_2$$

$$\int_1^2 \frac{\partial V}{\partial t}ds+\frac{V_2^{\,2}}{2}+\frac{p_2}{\rho}-F_2=\frac{V_1^{\,2}}{2}+\frac{p_1}{\rho}-F_1 \tag{44}$$

It should be pointed out that the integral in Eq.(44) is not always easy to evaluate.

References

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White, F.M. (2003). Fluid Mechanics, Fifth Edition, McGraw Hill Company, New York, NY.

Problems

1. The Reynolds transport theorem for the linear momentum **P** is given by

$$\left.\frac{D\mathbf{P}}{Dt}\right|_{SYS} = \frac{\partial}{\partial t}\int_{CV} \mathbf{V}\rho d\forall + \int_{CS} b\rho\mathbf{V}\cdot\mathbf{n}dA$$

where

$$\mathbf{P} = \int_{SYS} \mathbf{V}dm$$

From Newton's second law, the LHS term of the Reynolds transport theorem is

$$\left.\frac{D\mathbf{P}}{Dt}\right|_{SYS} = \sum\mathbf{F} \quad \text{(a)}$$

If one assumes that the CV is fixed with time, then one can write

$$\sum\mathbf{F} = \left.\frac{\partial\mathbf{P}}{\partial t}\right|_{CV} + \int_{CS} b\rho\mathbf{V}\cdot\mathbf{n}dA \quad \text{(b)}$$

which is valid at a time when the system coincides with the CV in a strict sense. Here, we have two ways of deriving the differential form of momentum equations. One is to use Eq.(a), by applying it to an infinitesimal small element of δm. The other is to use Eq.(b), similarly applying it to the CV of an infinitesimal small element of δm. Show that the two results are the same by using a 1D flow along a stream tube.

2. Derive the 1D continuity equation along the stream tube where the cross section varies, i.e., A to $A+dA$.

3. The 1D Euler equation along the streamtube (s-axis) can be written as

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial s}=-\frac{1}{\rho}\frac{\partial p}{\partial s}+f_s$$

Starting from the above equation, derive the following form of 1D momentum equation for steady flows:

$$d\left(\frac{p}{\gamma}+\frac{V^2}{2g}+z\right)=0$$

4. Derive the total derivative or material derivative such as

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$$

using the Taylor series expansion.

5. Starting from the 3D Euler equations such as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+g_x$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+g_y$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+g_z$$

Perform Reynolds decomposition and show that viscous terms due to turbulence appear in the turbulence-averaged equations.